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Effect of Dissipation and Dispersion on Slowly Varying Wavetrains

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A kinematic model for disturbance wave motions slowly modulated in space and time is developed, which describes the effects of amplitude and frequency dispersion, modal dependence, flow inhomogeneities, dissipation, and high-order wave dispersion and diffusion. The results, based on general variational principles, apply to waves in arbitrary continuous media, but are discussed primarily in the fluid-dynamic context. The more complete theory presented here reduces to several known wave models in various limits and reconciles, within the framework of a broader unifying approach, a number of differences found among existing, although more specialized, formulations. These latter models are surveyed and, in particular, we emphasize how their domains of applicability may be limited by different kinds of dominant high-order dispersive or diffusive wave mechanisms.

I. Introduction and Background

THE kinematics of wave propagation and stability, centrally important to the dynamics of disturbance systems in continuous media, no doubt involves the competing effects of amplitude and frequency dispersion, wave dissipation, background inhomogeneities, and high-order wave dispersion and diffusion. While many aspects of the foregoing interactions have been separately and successfully handled in numerous specific applications, no complete theory applicable to general nonlinear dissipative wave motions and all of its ramifications is yet available. Some significant progress toward a general wave theory has been made in recent years that draws upon the work of Hayes, Bretherton, Landahl, Whitham, Stewartson, and others; general conservation laws are postulated in terms of kinematical entities such as wavenumber, frequency, group velocity, and wave action. That a purely kinematic approach to wave mechanics is possible without specific reference to any particular set of governing equations is significant, because the basic elemental mechanisms and their interactions can be much better elucidated. The present paper re-examines several existing "low-order" wave models, points out their ranges of applicability, and, within the framework of a more complete and broader unifying approach, reconciles several inconsistencies and contradictions that have arisen in the literature. In this section we review several basic methods and their limitations. This survey places the contributions of the present paper in proper perspective.

Basic Kinematic Approach

The kinematic description of slowly varying wavetrains in conservative media was first initiated almost fifteen years ago by Whitham[1] as follows. Consider a physical system whose governing equation is derivable from a Lagrangian density L interpreted, say, as the difference between kinetic and potential energies. Evaluate L using the uniform plane wave solution (for instance, use $a\phi(y)\sin(Kx-\omega^r t)$ in the linear case), integrate over the modal coordinate y, if applicable, and average over the rapidly varying phase variable $\theta=Kx-\omega^r t$. Here a is an amplitude, K is the wavenumber, and ω^r is a real frequency. The resulting averaged Lagrangian $\mathcal{L}$ is defined over the space of a and θ and depends explicitly on a, K, and ω^r. A further parametric dependence on x and t reflects the presence of any nonuniformities in the supporting background medium. To connect coupled changes between a, K, and ω^r that occur over space and time scales large compared to a typical wavelength or period, Whitham allows the wave parameters to vary slowly and defines an averaged variational principle.[1] The Euler equations for these wave variables are obtained in the usual way. Variations with respect to a produce $\mathcal{L}_a(a,K,\omega^r,x,t)=0$ for the required phase or dispersion relation, while variations with respect to θ produce the so-called "action equation" $\partial\mathcal{L}_{\omega^r}/\partial t-\partial\mathcal{L}_K/\partial x=0$ for the wave action $A=\mathcal{L}_{\omega^r}$ (this can be viewed for the interim as an energy-like norm). The former Euler equation expresses energy equipartition in linear systems; for linear and nonlinear systems, this phase relation is identical to the dispersion relation of the uniform plane wave, only provided high-order modulations are excluded. The latter is an amplitude equation that nonlinearly couples with the phase relation through the postulated law for wave crest conservation $\partial K/\partial t+\partial\omega^r/\partial x=0$ where $\omega^r=\omega^r(a,K,x,t)$ is obtained by inverting $\mathcal{L}_a=0$. The dependences on x and t are assumed to be weak. The nonlinear coupling in the unsteady case is nontrivial and leads to interesting instabilities while, in the steady case, the required integrals are the anticipated constancies of action flux $\mathcal{L}_K$ and frequency ω^r. Various applications are discussed in Ref. 2. The simplest arises from the formulation for linear problems. In general, we can write $\mathcal{L}=a^2G(\omega^r,K,x,t)$ so that $\mathcal{L}_a=0$ gives $G=0$. Thus the action velocity $-\mathcal{L}_K/\mathcal{L}_{\omega^r}$ equals the wavenumber velocity ω^r_K (the wavenumber derivative is evaluated holding x and t fixed) and, furthermore, equals those for energy and momentum. In linear problems, ω^r_K is the well-known "group" velocity, but for nonlinear problems, because the former quantities generally propagate with unequal velocities, the "group" concept is less applicable. Whitham's equations uncouple in this linear limit: the wavenumber field can be determined first (because the frequency is independent of amplitude) and then used to yield $A(x,t)$. For both linear and nonlinear problems, the equivalence between the results generated by Whitham's simple and elegant approach and those low-order ones ob-

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tained using more conventional multiple-scaling and approximate WKBJ methods is amply demonstrated in Refs. 3-7 for a wide variety of problems. But Whitham's method leads immediately to a conservation law for the "adiabatic invariant" A because, apparently, the necessary canonical manipulations required in "direct" expansion methods are eliminated through the use of a more "compact" known variational formulation.

Whitham's low-order method applies to nonlinear waves but to conservative systems only. This latter restriction is severe, because all real systems are likely to involve some dissipation. Thus, Landahl[8] postulated the following change, but for linear systems only: First determine the wavenumber field using the real part of the complex frequency $\omega^c = \omega^r + i\omega^i$ which corresponds to the uniform wave, but in solving for the amplitude distribution, use the modified conservation law for wave action $\partial A/\partial t + (\partial \omega_K^r A)/\partial x = 2\omega^i A$.

Landahl's plausible modification for slightly nonconservative systems is formally justified in the present paper to include nonlinear wave motions and is extended to include high-order dispersive and diffusive corrections to both phase and amplitude laws. The utility of Landahl's idea is unquestionably important since it embodies Gaster's law relating temporal and spatial growth rates through the real group velocity. This forms the basis for many current transition analyses. The foregoing equations are also significant because they describe general wave propagation at the most fundamental level through descriptors like K, ω_K^r, ω^i, and A and are, therefore, useful in developing general notions on wave instability. This is a crucial application.

Some Inconsistencies and Apparent Contradictions

While Whitham's average Lagrangian approach to kinematic wave theory proved successful in numerous low-order applications, it gave high-order results inconsistent with those of more direct perturbation methods. For example, a WKBJ technique applied to the study of slow modulations of a Stokes wavetrain showed that higher-order terms representing modulation rates arising from the primary "linear" harmonic and not nonlinearity must be added to extend the time validity of Whitham's equations (see Ref. 9). The same ideas were re-emphasized by Davey,[10] who suggested various heuristic modifications, and by Davey and Stewartson,[11] who proceeded directly with the method of multiple scales (Whitham's results appear as a limit of these more accurate approaches). These difficulties arise even for strictly conservative systems: the high-order terms unaccounted for a Whitham's method are dispersive, and their neglect may be crucial. For instance, Whitham's equations suggest that a Stokes wave packet propagating over deep water eventually develops a singular cusplike energy distribution[12]; however, experiment[13] indicates that the large-time behavior is dominated by a train of solitons whose characteristics are correctly predicted by a "nonlinear Schroedinger equation," which can be derived by direct multiple-scaling expansions.

A secondary difficulty arises from extensions of Whitham's method to slightly nonconservative systems. Certainly all physical systems involve some dissipation, and Landahl's suggestions[8] should be meaningful in some sense. However, there are potential hazards. In many instances there is insufficient mathematical justification for attaching any degree of physical significance to, say, the real part of the complex group velocity. This point was highlighted in Stewartson's[14] criticism of Landahl's work, and some further work along Stewartson's lines appears in the mathematically motivated work of Gaster.[15] In fact, the results of the latter work have led Gaster himself to cast doubt on his well-known transformation[16] (we emphasize, however, the universal and successful use of Gaster's law in boundary-layer transition studies). In spite of this, Jimenez[17] and Jimenez and Whitham[18] have developed a nonlinear low-order theory for waves with slight dissipation (the model presented in this paper is a high-order one obtained somewhat differently, reducing correctly to the work of Ref. 18 in the appropriate limit). Analogous earlier studies were pursued by Lighthill,[19] Davey,[10] and Bretherton[20] and are more or less equivalent to Landahl's modification. Of course, waves in nonconservative media can be treated by direct two-scale methods at the expense of some algebraic complexity (e.g., see Ref. 21), but our objective is simplicity in the "averaged Lagrangian" sense.

Some Further Developments

Several additional references have appeared that further develop Whitham's average Lagrangian method and extend its diversity of application. We cite these for completeness. First note that Whitham's formalism[1,2] does not explicitly handle modal effects; in treating finite depth water waves[4] he uses a Lagrangian density of Luke[22] which appears as an integral of pressure over depth. A similar approach is employed by Gjevik[23] in his study of Love waves in elasticity. The modal formalism is important to a variety of waveguide applications; however, it was first given by Bretherton[24] for general linear wave systems and later extended to cover nonlinearity by Hayes.[25] Hayes' somewhat different approach furnished an additional conservation law for a new invariant entity in the modal coordinate space. Both treatments do not extend to high order and apply to nondissipative systems only (the low-order results of Jimenez and Whitham[18] for slightly nonconservative nonlinear systems do not include modal effects). Section III of this paper provides the necessary generalizations—we derive a high-order nonconservative model applicable to fully nonlinear modal problems and also an extended modal conservation law that is reducible to Hayes'.

An important application of wave action conservation appears in the study of wave motions through moving media, for example, surface waves on variable currents or plasma waves in nonuniform mean flows. A rather detailed investigation of Bretherton and Garrett[26] shows that action conservation, interpreted in the proper reference frame, leads directly to the "radiation stress" ideas advanced in a series of papers by Longuet-Higgins and Stewart.[27-29] The latter authors explore in detail the energy transfer and mutual work interaction between mean flow and wave. Their results, which are identical to those of Whitham[3] and Phillips,[30] who average the basic conservation laws directly, are reproduced by Bretherton and Garrett for gravity and capillary waves, and further extended for arbitrary linear waves in general moving media. These important results are extended in Sec. III to handle nonlinearity and dissipation using simpler physical arguments. Further applications of the variational method are cited in Ref. 2. In addition, variational formulations for a number of problems in continuum mechanics are presented in Ref. 31. However, we emphasize that while the variational formulation is desirable in the treatment of fully nonlinear waves, it can be completely bypassed for weak nonlinearity, even when high-order modulations are included. In the latter case it suffices to know the complex dispersion relation for the uniform plane wave and the Stokes correction to the real frequency (Sec. III).

The format of the remainder of this paper is as follows. Section II deals with the high-order consequences of the added modulation terms, but for near-linear problems. Diffusive corrections to a so-called "nonlinear Schroedinger equation" are introduced. Also, the Whitham-Landahl model is explained in the context of more general modulation equations; the retention of still higher-order terms lead to the model equations of Stewartson[14] and Stewartson and Stuart.[21] Section III extends the Lagrangian formalism to high order and includes dissipative effects. Section III also considers a number of practical applications related to wave stability.

Finally, in Sec. IV, some closing remarks are offered and some limitations of the kinematic approach are explained.

II. Structure of the High-Order Terms: Near-Linear Problems

Whitham's low-order results for conservative wave systems embody the usual notions on group velocity and energy propagation[2] (e.g., those described using classical stationary phase methods) and form the groundwork for a fairly general wave theory. Landahl's modifications, justified later, extend the range of applicability of Whitham's equations to include slightly dissipative systems. Low-order theory is generally used to uncover the singularities characterizing a real flow, however, and this may be questionable on mathematical grounds. For instance, the equation $u_t + uu_x = 0$ possesses shocklike solutions that suggest the presence of large gradients in the physical problem. In this case, the severe gradients exist only if the low-order equation arises as an approximation to the diffusive Burgers' equation $u_t + uu_x = \epsilon u_{xx}$ and not the dispersive Korteweg-deVries equation $u_t + uu_x = \epsilon u_{xxx}$ ($\epsilon = 0+$ in both cases). Similar considerations apply in problems for gasdynamic shocks.

The same problem appears in kinematic wave theory. While low-order methods are self-consistent within the framework of their formulation, it is not clear that their singularities actually identify with the rapid gradients of the real problem. This depends on the dispersive or diffusive nature of the high-order modulations; the foregoing examples, in fact, suggest that low-order wave focusing and shock formation are meaningful if the high-order terms are "more diffusive than dispersive"—but this is speculative. This difficult question, not considered in Landahl's early work because it entailed some complexity, was raised by Stewartson.[14] One objective here aims at clarifying the conditions under which the low-order model holds. In this section, we derive the general diffusive and dispersive structure of the high-order modulations for both amplitude and phase relations. The results are new and are used to resolve the issues discussed in Sec. I. The formulas given here are correct and are reducible to some results of Benney[32] obtained by a different method. The more general results presented in this section are obtained using Fourier-integral and multiple-scaling methods and are applicable to waveguide motions. However linearity is assumed and the effects of flow inhomogeneities present in the supporting background medium are neglected (these restrictions are removed in Sec. III). Of course the structure of these corrections holds for weakly nonlinear systems (such as were treated in Refs. 9 and 11) since, over those space-time scales in which weakly nonlinear effects are likely to become important, the form of the high-order modulations depends only on the primary or "linear" harmonic.

From a strictly mathematical point of view, the dispersion relation, which describes the entire family of plane wave solutions, must form the basis from which all general wave motions are constructed; for example, as through Fourier superposition. Thus, it is clear that this is all that is needed to describe the dynamics of any flow, and it remains for us to extract from it the physical laws governing those slowly varying wavetrains that are observed in the laboratory (e.g., the waves formed when a general disturbance asymptotically disperses into waves). The required construction proceeds first by expanding the modulated wave about a centered wavenumber K_0 and a real centered frequency Ω_0^R; second, by carrying out a physically motivated multiple-scaling analysis; and, third, by resumming the series results by analytic continuation in order to produce solutions valid for all wavenumbers (this last step is akin to replacing $1 + x + x^2 + \ldots$ with $(1-x)^{-1}$ which has a wider range of convergence). Thus we consider initially the superposition of monochromatic wave components like $\exp i(Kx - \Omega t)$, each satisfying the plane wave dispersion relation $\Omega(K) = \Omega^R + i\Omega^i$. (This superposition is similar to an elementary application of Lighthill.[19]) Here, again, K is a real wavenumber and Ω^R and Ω^i denote the real and imaginary parts of $\Omega(K)$. In the neighborhood of K_0, Taylor expansion gives

$$\Omega = \sum_{n=0}^{\infty} \frac{1}{n!} \Omega_{nK}(K_0)(K-K_0)^n$$

If $\Omega^i/\Omega^R \sim 0(\epsilon) \ll 1$, we can superpose elementary solutions to arrive at the more general integral

$$\tilde{F}(x,t) = \int_{-\infty}^{\infty} B(K) e^{i(Kx-\Omega t)} \mathrm{d}K$$

The function $B(K)$ is proportional to the Fourier transform of the initial condition and it exists when the initial disturbances are localized. Without loss of generality we write

$$\tilde{F}(x,t) = \psi(x,t) \exp[i(K_0 x - \Omega_0^R t)]$$

$$\psi = \int_{-\infty}^{\infty} B(K) \exp\{i[(K-K_0)x - (\Omega - \Omega_0^R)t]\} \mathrm{d}K \equiv \int_{-\infty}^{\infty} G \mathrm{d}K$$

so that $\tilde{F}(x,t)$ consists of a purely periodic part and an amplitude function $\psi(x,t)$ incorporating dissipative effects. If we apply the operator

$$\int_{-\infty}^{\infty} \ldots B(K) \exp i[(K-K_0)x - (\Omega - \Omega_0^R)t] \mathrm{d}K$$

to the Taylor-expanded dispersion relation, and note that

$$i\psi_t = \int_{-\infty}^{\infty} (\Omega - \Omega_0^R) G \mathrm{d}K$$

and

$$(-i)^n \psi_{nx} = \int_{-\infty}^{\infty} (K-K_0)^n G \mathrm{d}K$$

we obtain the identity

$$i\frac{\partial \psi}{\partial t} = i\Omega_0^i \psi + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \Omega_{nK,0} \frac{\partial^n \psi}{\partial x^n}$$

The analysis of Davey,[10] for example, keeps only two terms of the foregoing series; he implicitly assumes that wavenumber variations are less rapid than those in amplitude. Without any truncation we next seek WKBJ "wavelike" solutions of the form $\psi = a(X,T) \exp i\tilde{\theta}(x,t)$ where $a(X,T)$ is an auxiliary wave amplitude that depends explicitly on two "slow" variables $X = \epsilon x$ and $T = \epsilon t$ which measure space-time variations over scales much greater than those associated with a typical wavelength or period (the $\epsilon \ll 1$ indicates the relative weakness of nonconservative effects). A "fast" oscillatory scale can be introduced through the perturbation phase function $\tilde{\theta}(x,t)$ assumed real. We choose $\tilde{\theta}$ to satisfy $\tilde{\theta}_x = K(X,T) - K_0 = \mathcal{K}(X,T)$ and $-\tilde{\theta}_t = \omega(X,T) - \Omega_0^R = \tilde{\omega}(X,T)$, where ω here now represents the real frequency of the slowly varying wavetrain (in contrast to Ω^R, the real frequency of the uniform plane wave). Here, $\mathcal{K}$ and $\tilde{\omega}$ are quantities that satisfy the kinematic requirement for "wave crest conservation" $K_T + w_X = 0$, which, in effect, balances the local temporal wavenumber growth with fluxes in frequency. The multiple scaling chosen requires for any function $u(x,t) = U(\tilde{\theta}, X, T)$ that $u_x = \mathcal{K} U_{\tilde{\theta}} + \epsilon U_X$ and, in general, by induction,

$$u_{nx}(x,t)=\mathcal{K}^n U_{n\bar\theta}+\epsilon[n\mathcal{K}^{n-1}U_{(n-1)\bar\theta,X}+\tfrac{1}{2}N_1\mathcal{K}^{n-2}\mathcal{K}_X U_{(n-1)\bar\theta}]$$

$$+\epsilon\left[\frac{1}{2}N_1\mathcal{K}^{n-2}U_{(n-2)\bar\theta,XX}+\frac{1}{2}N_2\mathcal{K}^{n-3}\mathcal{K}_X U_{(n-2)\bar\theta,X}+\frac{1}{8}N_3\mathcal{K}^{n-4}\mathcal{K}_X^2 U_{(n-2)\bar\theta}+\frac{1}{6}N_2\mathcal{K}^{n-3}\mathcal{K}_{XX}U_{(n-2)\bar\theta}\right]$$

$$+\epsilon^3\left[\begin{array}{l}\frac{1}{12}N_4\mathcal{K}^{n-5}\mathcal{K}_X\mathcal{K}_{XX}U_{(n-3)\bar\theta}+\frac{1}{6}N_3\mathcal{K}^{n-4}\mathcal{K}_{XX}U_{(n-3)\bar\theta,X}+\frac{1}{4}N_3\mathcal{K}^{n-4}\mathcal{K}_X U_{(n-3)\theta,XX}+\frac{1}{6}N_2\mathcal{K}^{n-3}U_{(n-3)\bar\theta,XXX}\\ +\frac{1}{24}N_3\mathcal{K}^{n-4}\mathcal{K}_{XXX}U_{(n-3)\bar\theta}+\frac{1}{8}N_4\mathcal{K}^{n-5}\mathcal{K}_X^2 U_{(n-3)\bar\theta,X}+\frac{1}{48}N_5\mathcal{K}^{n-6}\mathcal{K}_X^3 U_{(n-3)\bar\theta}\end{array}\right]$$

$$+\ldots+\epsilon^n U_{nX}$$

where $N_j=n(n-1)(n-2)\ldots(n-j)$. The identity for ψ is now further expanded using this and the assumed Ansatz. Equating real and imaginary parts in the resulting equation gives, respectively,

$$a\bar\omega=\epsilon\sum_{n=1}^{\infty}\frac{1}{n!}\Omega^i_{nK,0}\left[n\mathcal{K}^{n-1}a_X+\frac{1}{2}N_1\mathcal{K}^{n-2}\mathcal{K}_X a\right]+a\sum_{n=1}^{\infty}\frac{1}{n!}\Omega^R_{nK,0}\mathcal{K}^n$$

$$-\epsilon^2\sum_{n=1}^{\infty}\frac{1}{n!}\Omega^R_{nK,0}\left\{\frac{1}{2}N_1\mathcal{K}^{n-2}a_{XX}+\frac{1}{2}N_2\mathcal{K}^{n-3}\mathcal{K}_X a_X+\frac{1}{8}N_3\mathcal{K}^{n-4}\mathcal{K}_X^2 a+\frac{1}{16}N_2\mathcal{K}^{n-3}\mathcal{K}_{XX}a\right\}$$

and

$$\epsilon a_T=\Omega^i_0 a-\epsilon\sum_{n=1}^{\infty}\frac{1}{n!}\Omega^R_{nK,0}\left[n\mathcal{K}^{n-1}a_X+\frac{1}{2}N_1\mathcal{K}^{n-2}\mathcal{K}_X a\right]a\sum_{n=1}^{\infty}\frac{1}{n!}\Omega^i_{nK,0}\mathcal{K}^n$$

$$-\epsilon^2\sum_{n=1}^{\infty}\frac{1}{n!}\Omega^i_{nK,0}\left\{\frac{1}{2}N_1\mathcal{K}^{n-2}a_{XX}+\frac{1}{2}N_2\mathcal{K}^{n-3}\frac{1}{2}\mathcal{K}_X a_X+\frac{1}{8}N_3\mathcal{K}^{n-4}\mathcal{K}_X^2 a+\frac{1}{6}N_2\mathcal{K}^{n-3}\mathcal{K}_{XX}a\right\}$$

$$+\epsilon^3\sum_{n=1}^{\infty}\frac{1}{n!}\Omega^R_{nK,0}\left\{\begin{array}{l}\frac{1}{12}N_4\mathcal{K}^{n-5}\mathcal{K}_X\mathcal{K}_{XX}a+\frac{1}{6}N_3\mathcal{K}^{n-4}\mathcal{K}_{XX}a_X+\frac{1}{6}N_2\mathcal{K}^{n-3}a_{XXX}\\ +\frac{1}{4}N_3\mathcal{K}^{n-4}\mathcal{K}_X a_{XX}+\frac{1}{24}N_3\mathcal{K}^{n-4}\mathcal{K}_{XXX}a+\frac{1}{8}N_4\mathcal{K}^{n-5}\mathcal{K}_X^2 a_X+\frac{1}{48}N_5\mathcal{K}^{n-6}\mathcal{K}_X^3 a\end{array}\right\}$$

The first is simply the "dispersion" or phase relation for the slowly varying wavetrain, while the second is a propagation law for wave amplitude. These apply to all wavenumbers. Summation of the foregoing series leads to a phase relation of the form

$$\omega=\Omega^R+\epsilon\Omega^i_K\frac{a_X}{a}+\frac{1}{2}\epsilon K_X\Omega^i_{KK}-\frac{1}{2}\epsilon^2\Omega^R_{KK}\frac{a_{XX}}{a}-\frac{1}{2}\epsilon^2\Omega^R_{3K}\frac{a_XK_X}{a}-\frac{1}{8}\epsilon^2\Omega^R_{4K}K_X^2-\frac{1}{6}\epsilon^2\Omega^R_{3K}K_{XX} \tag{1}$$

and, after some algebra, to the amplitude equation

$$\frac{\partial}{\partial T}a^2+\frac{\partial}{\partial X}\Omega^R_K a^2=2\frac{\Omega^i}{\epsilon}a^2-\epsilon^2\left\{aa_{XX}\frac{\Omega^i_{KK}}{\epsilon}+aa_XK_X\frac{\Omega^i_{3K}}{\epsilon}+\frac{1}{4}a^2K_X^2\frac{\Omega^i_{4K}}{\epsilon}+\frac{1}{3}a^2K_{XX}\frac{\Omega^i_{3K}}{\epsilon}\right\}$$

$$+\epsilon^2\left\{\frac{1}{3}aa_{XXX}\Omega^R_{3K}+\frac{1}{2}K_Xaa_{XX}\Omega^R_{4K}+\frac{1}{3}aa_XK_{XX}\Omega^R_{4K}+\frac{1}{12}a^2K_{XXX}\Omega^R_{4K}+\frac{1}{6}a^2K_XK_{XX}\Omega^R_{5K}+\frac{1}{4}K_X^2aa_X\Omega^R_{5K}+\frac{1}{24}a^2K_X^3\Omega^R_{6K}\right\} \tag{2}$$

Equations (1) and (2) are coupled through the kinematic requirement for wave crest conservation:

$$\frac{\partial K}{\partial T}+\frac{\partial\omega}{\partial X}=0 \tag{3}$$

Because $\Omega^i/\Omega^R\sim 0(\epsilon)$, all high-order corrections to the $\omega=\Omega^R$ part of Eq. (1) are $0(\epsilon^2)$. Similarly, all corrections to $(a^2)_t+(\Omega^R_K a^2)_x=2\Omega^i a^2$ in Eq. (2) are $0(\epsilon^2)$. Thus, for linear systems, Landahl's modification approximates the exact modulation equations to $0(\epsilon^2)$. The additional terms include those of Stewartson.[14] In general, these results show that the coefficients of all the differentiated corrections appearing in Eqs. (1) and (2) depend only on high-order wavenumber derivatives of the functions Ω^R and Ω^i (these are defined by the simple plane wave dispersion relation, of course). Again, Eqs. (1-3) hold for linear dissipative wavetrains in homogeneous media and indicate the role of the complex dispersion relation of normal mode theory in the dynamics of slowly varying waves.

Different limiting theories can be systematically developed. First consider linear waves. If all $0(\epsilon^2)$ corrections are dropped from the foregoing equations, we have $\partial K/\partial t + \partial \Omega^R(K)/\partial x = 0$ and $\partial a^2/\partial t + \partial \Omega_K^R a^2/\partial x = 2\Omega^i a^2$. The first leads to $K(x,t) = K_0(Z)$ where $Z = x - \Omega_K^R(K)t$ and $K_0 = K(x,0)$. The solution to the second equation is also easily obtained. For example, take $a(x,0) = a_0(x)$ and $\Omega^i = 0$ so that $a^2(x,t) = a_0^2(Z)/(1 + t\Omega_{KK}^R(K)K_0'(Z))$. Combination with the solution for K shows that, provided shocks do not form, a^2 decays inversely with t along a ray, agreeing with the results of classical stationary phase methods. Next, consider $\Omega^i \neq 0$ with $\partial/\partial x = 0$ so that $a^2 \sim \exp 2\Omega^i t$. Alternatively, for $\partial/\partial t = 0$, we have the solution $a^2 \sim [\Omega_K^R(K)]^{-1} \exp 2\int(\Omega^i/\Omega_K^R)\mathrm{d}x$ where the denominator $\Omega_K^R(K)$ is constant along a ray. These two formulas connect the temporal and spatial growth rates through the real group velocity Ω_K^R as required in Ref. 16, although, we note the results were obtained differently. Of course the more general modulation equations allow disturbance waves unsteady in both space and time. The $2\Omega^i a^2$ term was proposed in Ref. 8, and independently in Refs. 10 and 20. Here it is derived naturally using multiple-scaling models.

For nondissipative problems, the foregoing results extend to weak nonlinearity on using Stokes' correction with $\omega = \Omega^R(K) + a^2 h(K)$. This same $h(K)$ appears in the plane wave dispersion relation. To leading order, wave conservation implies that $\partial K/\partial t + \Omega_K^R \partial K/\partial x + h(K)\partial a^2/\partial x = 0$. The retained amplitude term corrects the characteristics to $0(a)$, while the neglected term introduces relative corrections of $0(a^2)$. To the same order, this couples with $\partial a^2/\partial t + \Omega_K^R \partial a^2/\partial x + \Omega_{KK}^R a^2 \partial K/\partial x = 0$. These are easily put in characteristic form (e.g., see Ref. 2). Then, for $h\Omega_{KK}^R > 0$, the wave packet "splits" and travels along two distinct characteristics (in linear theory, a wave distorts but does not split); here, the possibility of compressive modulations raises the question of multivalued solutions and breaking. For $h\Omega_{KK}^R < 0$, the modulation equations are elliptic, suggesting an ill-posed Cauchy problem: small perturbations grow with time and the periodic wavetrain is unstable.[2] This instability has been observed experimentally.[13] To extend these results to slightly dissipative systems, simply retain the $2\Omega^i a^2$ term of Eq. (2). This modification agrees with Jimenez and Whitham,[18] which appeared simultaneously with the present author's original work[33] (Ref. 18 does not, however, handle modal effects or high-order modulations). Note that nonconservative effects in linear systems do not affect the wavenumber distribution. This is untrue, however, of nonlinear systems. As an example, consider the propagation of gravity waves on a deep, but nonuniform, current $U(x)$, for which the modulation equations are elliptic. Three "nonconservative" effects enter the interaction: Whitham's instability, the dissipation Ω^i due to viscosity, and the energy transfer from mean flow to wave. Simple calculations[33] show than an additional exponential factor proportional to $\exp(\Omega^i - U_x)t$ modulates Whitham's amplitude growth.

Some simplified high-order wave models are also easily developed. First consider conservative systems. The former results, of course, hold, but in some applications, equally important high-order dispersive effects arise from the "linear" or primary harmonic that interact with Stokes' weak nonlinearity. As before, the most important modification applies to the phase relation. Assuming that 1) wavenumber variations are less rapid than those of amplitude or that 2) Ω_{3K}^R, Ω_{4K}^R,... are negligible, we have $\omega = \Omega^R(K) + h(K)a^2 - \frac{1}{2}\epsilon^2 \Omega_{KK}^R a_{XX}/a$. This was given in Ref. 14, although it does not cite 1 or 2. To this order, it is solved with Eq. (2) with the right side identically zero. Dropping the dispersive correction leads to low-order theory. Before extension to dissipative problems, we summarize some known consequences of the dispersive modification. The approximation is best understood by re-expanding both amplitude and phase relations about a center wavenumber K_0. Then,

$$(a^2)_t + ((C_0 + \Omega_{KK,0}^R \tilde{\theta}_X)a^2)_x = 0$$

and

$$\omega - \Omega_0^R = C_0(K - K_0) + \tfrac{1}{2}\Omega_{KK,0}^R(K - K_0)^2 + h(K_0)a^2 - \tfrac{1}{2}(\Omega_{KK,0}^R a_{xx}/a)$$

where $C_0 = \Omega_{K,0}^R$. The former notation $\psi = a\exp i\tilde{\theta}$ then leads to

$$i(\psi_t + C_0\psi_x) + \tfrac{1}{2}\Omega_{KK,0}^R \psi_{xx} = h(K_0)|\psi|^2\psi$$

This is the "nonlinear Schroedinger equation" (NSE) which arises in water waves and plasma physics. Further discussion is found in Ref. 34. Essentially, for initial conditions that approach zero rapidly at infinity, the NSE is soluble using exact inverse scattering methods.[35] An initial wave envelope pulse of arbitrary shape eventually disintegrates into "solitons," plus an unsteady oscillatory "tail." The number and structure of these solitons and the structure of the noisy tail are completely determined by initial conditions. The solitons are stable in the following sense—they survive interactions with each other without permanent change, except for a possible shift in position and phase. In general, the end product of Whitham's unstable "elliptic" modulations is a train of solitons. This conclusion is consistent with the experiments of Yuen and Lake[13]: Lighthill's[12] cusplike energy density distribution, obtained using Whitham's low-order equations, simply does not exist. The NSE, furthermore, is consistent with Chu and Mei.[9]

Systematic improvements are readily available through appropriate high-order analysis. Modifications for nonzero Ω^i have not been pursued in the literature, except to the extent of Benney's work, previously cited.[32] Within the framework of Eqs. (1) and (2), the required changes are trivial. The same reasoning leading to the former modified phase relation requires the retention of the dominant $\epsilon\Omega_K^i a_X/a$ term in Eq. (1). This balances an equally important dispersion term. At the same time, the $2\Omega^i a^2$ term in the amplitude equation must be kept. The resulting system generalizes that leading to the NSE for nonconservative effects. On re-expansion about $K = K_0$, a modified nonlinear equation for ψ can be obtained that is similar to an equation of Stewartson and Stuart.[21] Many boundary-layer flows are known to be only weakly dispersive.[8] In this limit, the dispersive term in the modified phase relation drops out. Because the high-order terms are now "more diffusive than dispersive," Landahl's use of low-order theory in predicting the breakdown of laminar shear flows may be justifiable, but no formal proofs are available. For still higher-order modulations, we include both the diffusion term $-\epsilon\Omega_{KK}^i a a_{XX}$ and the dispersion term $\epsilon^2\Omega_{3K}^R a a_{XXX}$ in the equation for a^2. Note that the latter term vanishes for weakly dispersive problems, leaving a "Burgers'-like" equation. This diffusion equation possibly furnishes a shock structure for any low-order singularities, and removes certain objections of Stewartson[14] and Gaster,[15] at least for nondispersive problems. In the next hierarchy of approximations, high-order terms involving spatial derivatives of wavenumber appear in the obvious manner.

The diffusion coefficient $-\Omega_{KK}^i(K)$ in Eq. (2) deserves special attention. For deep-water waves, the damping rate $\Omega^i \sim -\nu_0 K^2$ and $-\Omega_{KK}^i$ is proportional to the kinematic viscosity ν_0—the satisfying presence of a "wave diffusion" term is consistent with the diffusive effect of real viscosity. But it is also possible to define a diffusion coefficient $-\Omega_{KK}^i(K)$ for perfectly inviscid flows, e.g., as through the solution of Rayleigh's stability equation.[36] Thus, "wave diffusion" generally exists regardless of real viscosity; moreover, it need not be positive. This raises the possibility of "anticascading." In a damped system with $\Omega^i(K) < 0$, a range of wavenumbers K admitting negative diffusion would allow a large-time energy transfer back into the damped spectrum.

For typical boundary-layer velocity profiles, Orr-Sommerfeld solutions generally produce positive $-\Omega^i_{KK}$'s (e.g., see Fig. 9 of Ref. 8). However, for warm electron plasma waves, which are classically Landau-damped,[37] the diffusion coefficient is negative for a restricted wavenumber range; some numerical calculations pursued along these lines would be extremely interesting. These ideas are speculative, of course. Finally, note that the nonlinear parabolic equation of Stewartson and Stuart,[21] previously referenced, leads to possible instabilities; for certain classes of initial disturbances the amplitude distributions exhibit a "finite time" infinite peak at the wave group center.[38] The results of this section bear the limiting restrictions for near-linearity and, in addition, do not apply in the presence of background inhomogeneity. These restrictions are removed by introducing the more general variational approach pursued in the next section.

III. General Variational Formalism

The simple harmonic approach does not handle inhomogeneities (i.e., variable coefficients) or strong nonlinearities. To obtain more general results, we assume that the physical system is derivable from a variational principle with Lagrangian density L and dissipation function F. If $\epsilon \ll 1$ characterizes a slow variation proportional to the weakness of typical nonconservative effects, we can write

$$e_{\alpha,\beta,\gamma} \frac{\partial^{\alpha+\beta+\gamma}}{\partial x^\alpha \partial t^\beta \partial y^\gamma} \tilde{L} + L^* = \epsilon F(u,\dots)$$

$$L = L(\epsilon x, y, \epsilon t, u, u_{\alpha x,\beta t,\gamma y})$$

$$\tilde{L} = L_{u_{\alpha x,\beta t,\gamma y}} \quad L^* = L_u \quad e_{\alpha,\beta,\gamma} = (-1)^{\alpha+\beta+\gamma} \tag{4}$$

Here x,y, and t are the propagation, modal, and time coordinates. Also, α, β, and γ indicate differentiations over x, t, and y, where sums over the Greek letters are understood. For $F=0$, we assume the system admits propagating waves. Following Sec. II we seek slowly varying wave solutions of the form

$$u(x,y,t) = U(\theta, X, T, y; \epsilon)$$

$$X = \epsilon x \quad T = \epsilon t \quad \theta(x,t) = \epsilon^{-1}\Theta(X,T)$$

$$\omega(X,T) = -\theta_t = -\Theta_T = -\nu(X,T) \quad K(X,T) = \theta_x = \Theta_X \tag{5}$$

Again, X and T are stretched coordinates describing 0(1) changes in the dependent variable U over scales large in comparison with a typical wavelength or period, θ is a periodic phase function and, as before, the wavenumber K and the real frequency ω satisfy Eq. (3). We introduce the operator

$$\begin{aligned} H = {} & \nu^\beta K^\alpha \frac{\partial^{\alpha+\beta}}{\partial\theta^{\alpha+\beta}} + \epsilon\alpha\nu^\beta K^{\alpha-1} \frac{\partial^{\alpha+\beta}}{\partial\theta^{\alpha+\beta-1}\partial X} \\ & + \frac{1}{2}\epsilon\alpha(\alpha-1)\nu^\beta K^{\alpha-2} K_X \frac{\partial^{\alpha+\beta-1}}{\partial\theta^{\alpha+\beta-1}} + \epsilon\alpha\beta\nu^{\beta-1}K^{\alpha-1}K_T \frac{\partial^{\alpha+\beta-1}}{\partial\theta^{\alpha+\beta-1}} \\ & + \frac{1}{2}\epsilon\beta(\beta-1)\nu^{\beta-2}\nu_T K^\alpha \frac{\partial^{\alpha+\beta-1}}{\partial\theta^{\alpha+\beta-1}} + \epsilon\beta\nu^{\beta-1}K^\alpha \frac{\partial^{\alpha+\beta}}{\partial\theta^{\alpha+\beta-1}\partial T} \end{aligned} \tag{6}$$

Neglecting $0(\epsilon^2)$ terms, the foregoing Euler equation becomes

$$e_{\alpha,\beta,\gamma} \frac{\partial^\gamma}{\partial y^\gamma} H(\tilde{L}) + L^* = \epsilon F(U,\dots)$$

which, on applying the identity

$$\frac{\partial L}{\partial\theta} = L^* U_\theta + \tilde{L}\left(\frac{\partial^\gamma}{\partial y^\gamma} H(U)\right)_\theta$$

leads to

$$e_{\alpha,\beta,\gamma} U_\theta \frac{\partial^\gamma}{\partial y^\gamma} H(\tilde{L}) - (\tilde{L})\left(\frac{\partial^\gamma}{\partial y^\gamma} H(U)\right)_\theta + \frac{\partial L}{\partial\theta} = \epsilon U_\theta F(U) \tag{7}$$

Equation (7) will be used to construct a general action law valid for arbitrary continuous media. Two applications of the differential identity

$$f^{(n)} g = \left(\sum_{k=1}^{n} (-1)^{k+1} f^{(n-k)} g^{(k-1)}\right)' + (-1)^n f g^{(n)}$$

to Eq. (7)—first, with primes denoting θ derivatives (where $g = U_\theta$ and $f = \tilde{L}_{\gamma y}$) and second, with primes denoting y derivatives gives

$$\begin{aligned} & \partial/\partial\theta(L + e_{\alpha,\beta,\gamma}\Phi) = \epsilon U_\theta F(U) \\ & + \epsilon\Big\{\beta K^\alpha \nu^{\beta-1}\tilde{L}_T U_{(\alpha+\beta)\theta,\gamma y} + \beta(\beta-1)K^\alpha\nu^{\beta-2}\nu_T \tilde{L} U_{(\alpha+\beta)\theta,\gamma y} + \beta K^\alpha \nu^{\beta-1}\tilde{L} U_{\gamma y,(\alpha+\beta)\theta,T} + \alpha\beta K^{\alpha-1}K_T \nu^{\beta-1}\tilde{L} U_{(\alpha+\beta)\theta,\gamma y}\Big\} \\ & + \epsilon\Big\{\alpha K^{\alpha-1}\nu^\beta \tilde{L}_X U_{(\alpha+\beta)\theta,\gamma y} + \alpha(\alpha-1)K^{\alpha-2}K_X\nu^\beta \tilde{L} U_{(\alpha+\beta)\theta,\gamma y} + \alpha K^{\alpha-1}\nu^\beta \tilde{L} U_{\gamma y,(\alpha+\beta)\theta,X} + \alpha\beta K^{\alpha-1}\nu^{\beta-1}\nu_X \tilde{L} U_{(\alpha+\beta)\theta,\gamma y}\Big\} \\ & + (-1)^\gamma \frac{\partial}{\partial y}\left\{\begin{aligned} & -K^\alpha\nu^\beta\{\tilde{L}_{(\gamma-1)y}U_{(\alpha+\beta+1)\theta} - \tilde{L}_{(\gamma-2)y}U_{(\alpha+\beta+1)\theta,y} + \dots\} + \epsilon\beta K^\alpha \nu^{\beta-1}\{\tilde{L}_{T,(\gamma-1)y}U_{(\alpha+\beta)\theta} - \tilde{L}_{T,(\gamma-2)y}U_{(\alpha+\beta)\theta,y} + \dots\} \\ & + \epsilon\alpha K^{\alpha-1}\nu^\beta\{\tilde{L}_{X,(\gamma-1)y}U_{(\alpha+\beta)\theta} - \tilde{L}_{X,(\gamma-2)y}U_{(\alpha+\beta)\theta,y} + \dots\} + \epsilon\{\tilde{L}_{(\gamma-1)y}U_{(\alpha+\beta)\theta} - \tilde{L}_{(\gamma-2)y}U_{(\alpha+\beta)\theta,y} + \dots\} \\ & \times\left\{\frac{\alpha(\alpha-1)}{2}K^{\alpha-2}K_X\nu^\beta + \frac{\beta(\beta-1)}{2}K^\alpha\nu^{\beta-2}\nu_T + \alpha\beta K^{\alpha-1}K_T\nu^{\beta-1}\right\} \end{aligned}\right\} \end{aligned}$$

where

$$\begin{aligned} \Phi = {} & K^\alpha\nu^\beta\{\tilde{L}_{\gamma y,(\alpha+\beta-1)\theta}U_\theta - \tilde{L}_{\gamma y,(\alpha+\beta-2)\theta}U_{\theta\theta} + \dots\} + \epsilon\alpha K^{\alpha-1}\nu^\beta\{\tilde{L}_{\gamma y,(\alpha+\beta-2)\theta,X}U_\theta - \tilde{L}_{\gamma y,(\alpha+\beta-3)\theta,X}U_{\theta\theta} + \dots\} \\ & + \epsilon\beta K^\alpha\nu^{\beta-1}\{\tilde{L}_{\gamma y,(\alpha+\beta-2)\theta,T}U_\theta - \tilde{L}_{\gamma y,(\alpha+\beta-3)\theta,T}U_{\theta\theta} + \dots\} + \epsilon\left\{\frac{\alpha(\alpha-1)}{2}K^{\alpha-2}K_X\nu^\beta + \alpha\beta K^{\alpha-1}K_T\nu^{\beta-1} + \frac{\beta(\beta-1)}{2}K^\alpha\nu^{\beta-2}\nu_T\right\} \\ & \times\{\tilde{L}_{\gamma y,(\alpha+\beta-2)\theta}U_\theta - \tilde{L}_{\gamma y,(\alpha+\beta-3)\theta}U_{\theta\theta} + \dots\} \end{aligned}$$

Now, if L, $\bar{L}$, U, and Φ are expanded in series of the form

$$S=\sum_{n=0}^{\infty}\epsilon^n S^{(n)}(\theta,X,T)$$

and the results substituted in the foregoing equation, the leading terms obtained by equating coefficients in powers of ϵ give

$$-\frac{\partial}{\partial\theta}(L^{(0)}+e_{\alpha,\beta,\gamma}\Phi^{(0)})=\frac{\partial}{\partial y}K^\alpha\nu^\beta(-1)^\gamma\{\bar{L}^{(0)}_{(\gamma-1)y}U^{(0)}_{(\alpha+\beta+1)\theta}$$
$$-\bar{L}^{(0)}_{(\gamma-2)y}U^{(0)}_{(\alpha+\beta+1)\theta,y}+\ldots\}$$

Averaging over phase produces our first result, the "modal conservation law,"

$$\frac{\partial}{\partial y}\left[\frac{1}{2\pi}\int_0^{2\pi}(-1)^\gamma K^\alpha\nu^\beta\{\bar{L}^{(0)}_{(\gamma-1)y}U^{(0)}_{(\alpha+\beta+1)\theta}-\bar{L}^{(0)}_{(\gamma-2)y}U^{(0)}_{(\alpha+\beta+1)\theta,y}+\ldots\}\mathrm{d}\theta\right]=0 \tag{8}$$

which again holds for weakly nonconservative nonlinear systems with variable coefficients. The bracketed quantity, therefore, varies only slowly with space and time. Equation (8) extends Hayes'[25] modal law to differential equations of arbitrary order. The $0(\epsilon)$ equation leads to much more interesting results. First, introduce an average Lagrangian with the definition

$$\bar{L}=\frac{1}{2\pi}\int_0^{2\pi}L(X,T,y,U,H(U_{\gamma y}))\mathrm{d}\theta \tag{9}$$

Then, some algebra[33] shows that the result of phase and modal averages, assuming zero work interaction at the boundary (y_1,y_2), gives

$$\frac{\partial}{\partial T}\mathfrak{L}^{(0)}_\omega-\frac{\partial}{\partial X}\mathfrak{L}^{(0)}_K=\frac{1}{2\pi}\int_0^{2\pi}\int_{y_1}^{y_2}U^{(0)}_\theta F(U^{(0)})\mathrm{d}y\mathrm{d}\theta \tag{10}$$

where

$$\mathfrak{L}=\int_{y_1}^{y_2}\bar{L}\mathrm{d}y \tag{11}$$

These results easily extend to higher space dimensions with or without additional phase variables.

At this point, we summarize our findings for slowly varying wavetrains. The evolutionary equations are:

$$\frac{\partial}{\partial T}\mathfrak{L}^{(0)}_\omega-\frac{\partial}{\partial X}\mathfrak{L}^{(0)}_K=\frac{1}{2\pi}\int_0^{2\pi}\int_{y_1}^{y_2}U^{(0)}_\theta F(U^{(0)})\mathrm{d}y\mathrm{d}\theta\equiv\mathfrak{F}$$
$$\mathfrak{L}^{(0)}_a=0;\quad U(\theta,X,T,y;\epsilon)=U^{(0)}(a,\theta,y)$$
$$\partial K/\partial T+\partial\omega/\partial X=0 \tag{12}$$

When the Lagrangian density for a particular problem is evaluated using its plane wave solution, the first relation of Eq. (12) provides a conservation law for $\mathfrak{L}^{(0)}_\omega=A$. Then $\mathfrak{L}^{(0)}_a=0$ inverts to give a phase relation $\omega=\omega(K,a,X,T)$ *independent of dissipation and identical with the plane wave dispersion relation* which then couples with the dissipative action equation via Eq. (3) through wave conservation. These results apply to fully nonlinear equations of arbitrary order with variable coefficients and in the presence of weak dissipation. They are correct to leading order only. The group velocity concept is still relevant to weakly nonconservative systems, as is the notion of wave action—both definitions are furnished from the "conservative part" of the solution. Unlike the approach followed by Landahl,[8] we have completely bypassed the use of complex frequencies. Note that the first and last equations of Eq. (12) are nonlinearly coupled and are of mixed hyperbolic-elliptic character. It is also possible to cast the first of Eq. (12) in either of the following forms using Eq. (3)

$$\frac{\partial}{\partial T}(\omega\mathfrak{L}^{(0)}_\omega-\mathfrak{L}^{(0)})-\frac{\partial}{\partial X}(\omega\mathfrak{L}^{(0)}_K)=-\mathfrak{L}^{(0)}_T+\omega\overline{U^{(0)}_\theta F(U^{(0)})} \tag{13}$$

$$\frac{\partial}{\partial T}(K\mathfrak{L}^{(0)}_\omega)-\frac{\partial}{\partial X}(K\mathfrak{L}^{(0)}_K-\mathfrak{L}^{(0)})=\mathfrak{L}^{(0)}_X+\overline{KU^{(0)}_\theta F(U^{(0)})} \tag{14}$$

where bars indicate phase and modal integrations. Equation (13) is an energy law in the mathematical sense because energy is that quantity conserved when the variational principle is invariant to time translations [alternatively, we can derive Eq. (13) by considering an identity for $\partial L/\partial T$ in the work just following Eq. (6)]. Spatial translations lead to a momentum law given in Eq. (14). These equations generalize Whitham's[1,2] for dissipation and modal effects. They indicate the relative roles played by inhomogeneity and dissipation and show that only in linear systems (when $\mathfrak{L}^{(0)}=0$) are the velocities for wavenumber, action, energy, and momentum equal.

Higher-order extensions are obtainable by considering the three-variable variational principle

$$\delta\int_R\iint_0^{2\pi}L(X,T,U,H(U))\mathrm{d}\theta\mathrm{d}X\mathrm{d}T$$
$$-\epsilon\int_R\iint_0^{2\pi}F(U)\delta U\mathrm{d}\theta\mathrm{d}X\mathrm{d}T=0 \tag{15}$$

for the three-variable function $U(\theta,X,T)$, where U and its variations are periodic in θ, and variations in U vanish on the boundary R. Variation of U gives after manipulation the multiple-scaled Euler equation Eq. (7); the two-timed form of Eq. (4) is precisely the Euler equation obtained from the two-timed variational principle. It follows that Eq. (15) is exact because it contains the whole expansion; moreover, it is already in averaged form. Now, introduce an amplitude measure "a" as in Sec. II so that $U(\theta,X,T;\epsilon)$ depends explicitly on the parameters "a" and θ. Then, since

$$\delta\int_R\int\bar{L}\mathrm{d}X\mathrm{d}T=\epsilon\int_R\iint\left\{\frac{1}{2\pi}\int_0^{2\pi}F(U)(U_\theta\delta\theta+U_a\delta a)\mathrm{d}\theta\right\}\mathrm{d}X\mathrm{d}T$$

must hold, a variations lead to

$$\bar{L}_a-\frac{\partial}{\partial T}\bar{L}_{a_T}-\frac{\partial}{\partial X}\bar{L}_{a_X}+\ldots-\epsilon\frac{1}{2\pi}\int_0^{2\pi}U_aF(U)\mathrm{d}\theta=0 \tag{16}$$

while θ variations lead to

$$\frac{\partial}{\partial T}\bar{L}_\nu+\frac{\partial}{\partial X}\bar{L}_K-\frac{\partial^2}{\partial T^2}\bar{L}_{\nu_T}-\frac{\partial^2}{\partial X\partial T}\bar{L}_{K_T}$$
$$-\frac{\partial^2}{\partial X^2}\bar{L}_{K_X}+\ldots=-\frac{1}{2\pi}\int_0^{2\pi}U_\theta F(U)\mathrm{d}\theta \tag{17}$$

Equations (16) and (17) formally extend Eq. (12) to high order; these results are new and generalize Eqs. (1) and (2). As in Sec. II, high-order corrections to phase appear before any modifications to the basic amplitude law are made. Assuming

$F=0$, this amounts to retaining only the three terms of Eq. (16) shown and only the first two of Eq. (17). This is confirmed by the independent work of Yuen and Lake[13] who, for weakly nonlinear gravity waves in deep water, recover the NSE. Of course, Eqs. (16) and (17) are applicable to fully nonlinear applications. Thus, Whitham's Lagrangian method produces the correct high-order dispersive modulations if carried out consistently. On the other hand, the diffusion terms in Eqs. (1) and (2) appear naturally when the F integrals of Eqs. (16) and (17) are expanded out in linear examples. In either case, the plane wave solution used is the conservative one; the equations derived for "a" automatically account for nonzero F's *without* the need for complex frequencies (see Example 1).

Example 1. Linear Waves without Modal Structure

Consider the general linear system

$$\sum_\alpha \sum_\beta a_{\alpha,\beta}(\epsilon x,\epsilon t)\frac{\partial^{\alpha+\beta}u(x,t)}{\partial x^\alpha \partial t^\beta} = \epsilon \sum_\sigma \sum_\gamma b_{\sigma,\gamma}(\epsilon x,\epsilon t)\frac{\partial^{\sigma+\gamma}u}{\partial x^\sigma \partial t^\gamma} \quad (18)$$

where the sums are assumed over $\alpha+\beta=$ even and $\sigma+\gamma=$ odd. Conventionally one takes $u(x,t)\sim \exp i(Kx-\omega t)$ where $\omega=\omega^R+i\omega^i$ and $\omega^i/\omega^R \sim 0(\epsilon)$. To $0(\epsilon)$

$$\sum_\alpha \sum_\beta a_{\alpha,\beta}(-1)^\beta i^{\alpha+\beta}K^\alpha[\omega_r^\beta + i\beta\omega_r^{\beta-1}\omega_i]$$

$$= \epsilon \sum_\sigma \sum_\gamma b_{\sigma,\gamma}(-1)^\gamma i^{\sigma+\gamma}K^\sigma \omega_r^\gamma$$

Then, equating real and imaginary parts gives

$$\sum_\alpha \sum_\beta a_{\alpha,\beta}(-1)^\beta i^{\alpha+\beta}K^\alpha \omega_r^\beta = 0 \quad (19)$$

$$\frac{\omega_i}{\epsilon} = \frac{\sum_\sigma \sum_\gamma b_{\sigma,\gamma}(-1)^\gamma i^{\sigma+\gamma}K^\sigma\omega_r^\gamma}{\sum_\alpha \sum_\beta a_{\alpha,\beta}(-1)^\beta i^{\alpha+\beta+1}\beta K^\alpha \omega_r^{\beta-1}} \sim O(1) \quad (20)$$

Instead, we assume the left side of Eq. (18) to be derivable through

$$\sum_\alpha \sum_\beta a_{\alpha,\beta}\frac{\partial^{\alpha+\beta}u}{\partial x^\alpha \partial t^\beta} \equiv L_u - \frac{\partial}{\partial x}L_{u_x} - \frac{\partial}{\partial t}L_{u_t} + \frac{\partial^2}{\partial x^2}L_{u_{xx}} + \ldots$$

For convenience, introduce the normalization $a_{0,0}=1$ so that $L=\ldots+\frac{1}{2}u^2$. This, along with Eq. (19), shows that

$$\bar{L}_\nu^{(0)}(K,\nu,a^2) = \frac{1}{4}a^2 \sum_\alpha \sum_\beta a_{\alpha,\beta} i^{\alpha+\beta}\beta K^\alpha \nu^{\beta-1}$$

where L is evaluated using the *conservative* wave solution $U^{(0)}=a(X,T)\sin\theta$. Evaluating the right side of Eq. (18) with the same plane wave gives, successively,

$$\frac{\partial}{\partial T}\bar{L}_\nu^{(0)} + \frac{\partial}{\partial X}\bar{L}_K^{(0)} = -\frac{1}{2\pi}\int_0^{2\pi} \sum_\sigma \sum_\gamma b_{\sigma,\gamma}U_\theta^{(0)}U_{(\sigma+\gamma)\theta}^{(0)}K^\sigma \nu^\gamma \mathrm{d}\theta$$

$$= -\frac{1}{2}a^2 \sum_\sigma \sum_\gamma b_{\sigma,\gamma} i^{\sigma+\gamma-1}K^\sigma \nu^\gamma$$

$$= -2\bar{L}_\nu^{(0)} \frac{\sum_\sigma \sum_\gamma b_{\sigma,\gamma} i^{\sigma+\gamma-1}K^\sigma \nu^\gamma}{\sum_\alpha \sum_\beta a_{\alpha,\beta} i^{\alpha+\beta}(-1)^{\beta-1}\beta K^\alpha \omega_r^{\beta-1}}$$

$$= 2\bar{L}_\nu^{(0)}\frac{\omega_i}{\epsilon} \quad (21)$$

This is precisely Landahl's[8] modified action law; to the same order, Eq. (19) indicates that dissipation does not affect the phase. The use of complex frequencies is not strictly necessary. When the "conservative wave" Ansatz $U=a\sin\theta$ is used to evaluate the right side of Eq. (12), a dissipative law for the wave amplitude "a" appears naturally. This is crucial to many nonlinear problems where complex frequencies cannot be used.

Example 2. Linear Modal Waves

Modal waves involve a y structure orthogonal to the propagation direction x; e.g., acoustic waves in waveguides or Tollmien-Schlichting waves in a boundary layer. Consider

$$\sum_\alpha \sum_\beta \sum_\gamma a_{\alpha,\beta,\gamma}(\epsilon x,\epsilon t,y)\frac{\partial^{\alpha+\beta+\gamma}u(x,y,t)}{\partial x^\alpha \partial t^\beta \partial y^\gamma}$$

$$= \epsilon \sum_\sigma \sum_\tau \sum_\zeta b_{\sigma,\tau,\zeta}(\epsilon x,\epsilon t,y)\frac{\partial^{\sigma+\tau+\zeta}u}{\partial x^\sigma \partial t^\tau \partial y^\zeta} \quad (22)$$

where the left side is conservative and the right side is dissipative. The local plane wave may be obtained as the solution to

$$\sum_{\alpha\beta\gamma} a_{\alpha,\beta,\gamma}(0,0,y)\frac{\partial^{\alpha+\beta+\gamma}u}{\partial x^\alpha \partial t^\beta \partial y^\gamma} = \epsilon \sum_{\sigma\tau\zeta} b_{\sigma,\tau,\zeta}(0,0,y)\frac{\partial^{\sigma+\tau+\zeta}u}{\partial x^\sigma \partial t^\tau \partial y^\zeta} \quad (23)$$

Equation (23) has the eigenfunction $\bar{U}_s=\varphi_s(y)\exp i(Kx-\omega t)$ with mode number s and where $\varphi_s(y)$ is, in turn, determined from

$$\sum_{\alpha\beta\gamma} a_{\alpha,\beta,\gamma}(0,0,y)\varphi_{\gamma y}(y)(iK)^\alpha(-i\omega)^\beta$$

$$= \epsilon \sum_{\sigma\tau\zeta} b_{\sigma,\tau,\zeta}(0,0,y)\varphi_{\zeta y}(y)(iK)^\sigma(-i\omega)^\tau \quad (24)$$

plus appropriate homogeneous boundary conditions (the right side may be dropped if the undamped system has no critical layers). Then, in Eq. (12) the plane wave solution used is:

$$U_s^{(0)}(\theta,X,T,y) = a_s(X,T)\varphi_s(y)\sin\theta \quad (25)$$

and the derived equation for $a_s(X,T)$ accounts automatically for damping. The "shape invariance" implicit in Eq. (24) is physically justifiable provided the waves are, in fact, "slowly varying." Ideas similar to Eq. (25) appear in Refs. 39 and 40.

Example 3. Waves on Moving Currents

Equation (12) holds for general inhomogeneous media, but specialized results applicable to waves on slowly varying currents are easily derived which describe the mutual work interaction through the so-called "radiation stresses."[27-29] Local refractive changes to ω and K are, of course, accounted for through Eq. (3). Consider waves propagating parallel to the mean flow $U=U(X,T)$. If primes refer to coordinates S' moving with the mean flow relative to a fixed frame S, Galilean invariance requires that $\mathcal{L}^{(0)}(X,T,\omega,K)=\mathcal{L}'^{(0)}(X',T',\omega',K)$ where $\omega'=\omega-UK$. This contains the expected Doppler shift. That the fictitious forces experienced in S' may be interpreted as equivalent Reynolds stresses as observed from S motivates us to define energy densities $E'=\omega'\mathcal{L}'^{(0)}_{\omega}-\mathcal{L}'^{(0)}$ and $E=\omega\mathcal{L}_\omega^{(0)}-\mathcal{L}^{(0)}$ [see Eq. (13)]. Then the invariance $\mathcal{L}_\omega^{(0)}=\mathcal{L}'^{(0)}_{\omega}=\mathcal{L}_\omega^*$ implies that $(E'+\mathcal{L}'^{(0)})/\omega'=(E+\mathcal{L}^{(0)})/\omega=\mathcal{L}_\omega^*$. For linear systems, we have $\mathcal{L}^{(0)}=\mathcal{L}'^{(0)}=0$. Substitution of $\mathcal{L}_\omega^{(0)}=E'/\omega'$ in $\partial\mathcal{L}_\omega^{(0)}/\partial t-\partial\mathcal{L}_K^{(0)}/\partial x=2\omega^i\mathcal{L}_\omega^{(0)}$, and simplification using Eq. (3), leads to

$$\frac{\partial E'}{\partial t} + \frac{\partial}{\partial x}(U+\Omega_K)E' = E'\left[\frac{\Omega_t + U\Omega_x - K\Omega_K U_x}{\Omega} + 2\omega_i\right] \quad (26)$$

where we have denoted $\Omega(K,X,T) = \omega'$. For both gravity and capillary waves in water of finite depth, this formula produces the correct energy interaction.[26,33] It also holds for more general fluid motions—when the Ω in $\omega = \Omega + UK$ is given, the right side of Eq. (26) directly gives the interplay between radiation stresses and dissipation. The foregoing derivation simplifies the rather involved arguments of Ref. 26, extends their results to dissipative media, but, more importantly, enables a ready extension to nonlinear dissipative problems (see below).

Again, for linear systems an analogous wave momentum law is easily derived. The energy and momentum densities E' and M' are related through the relative phase velocity by $E'/\Omega = M'/K$. Use of Eqs. (3) and (26) leads to

$$\frac{\partial M'}{\partial t} + \frac{\partial}{\partial X}(U+\Omega_K)M' = -M'\left[\frac{\partial U}{\partial X} + \frac{\Omega_x}{K} - 2\omega_i\right] \quad (27)$$

This result appears to be new and is not given in Ref. 26. Both Eqs. (26) and (27) can be extended to cover full nonlinearity. The foregoing ideas on Galilean invariance applied to Eqs. (13) and (14) lead directly to

$$\frac{\partial E'}{\partial T} + \frac{\partial}{\partial X}\left(U - \frac{\Omega\mathcal{L}'_K}{\Omega\mathcal{L}'_\Omega - \mathcal{L}'}\right)E' = (K\mathcal{L}'_K - \mathcal{L}')\frac{\partial U}{\partial X} - (\mathcal{L}'_T + U\mathcal{L}'_X) + \Omega\mathfrak{F} \quad (28)$$

$$\frac{\partial M'}{\partial T} + \frac{\partial}{\partial X}\left(U + \frac{\mathcal{L}' - K\mathcal{L}'_K}{K\mathcal{L}'_\Omega}\right)M' = \mathcal{L}'_X - M'\frac{\partial U}{\partial X} + K\mathfrak{F} \quad (29)$$

These results also appear to be new. Equations (28) and (29) are solved with Eq. (3) using an appropriate nonlinear dispersion relation. Note that energy and momentum propagate with generally unequal velocities since $\mathcal{L}'$ is nonzero. Integrated wave properties are readily determined. Consider the net energy

$$\Sigma(t) = \int_{x_1}^{x_2} E'(x,t)\,\mathrm{d}x$$

between two rays. For example, in the linear case, Eq. (26) holds; hence,

$$\frac{\mathrm{d}\Sigma(t)}{\mathrm{d}t} = \int_{x_1(t)}^{x_2(t)} E'(x,t)\left[\frac{\Omega_t + U\Omega_x - K\Omega_K U_x}{\Omega} + 2\omega_i\right]\mathrm{d}x \quad (30)$$

$\Sigma(t)$ is easily determined once all local properties are known. In general, the net energy may decrease while the energy density becomes infinite at some point and conversely. Similar considerations apply to net momentum. Further discussion and examples are available in Ref. 33.

IV. Discussion and Conclusions

The primary contribution of the present paper is a unifying formulation which encompasses several known wave models. For linear and weakly nonlinear waves we gave the general form of the high-order terms (Sec. II). These corrections are both dispersive and diffusive; their effects generally dominate those of low-order theory over large space and time scales. Also, we showed how accepted high-order dispersive modifications to phase[10,11,14,34] and high-order diffusive modifications to the amplitude equation[21] arise naturally from one simple "baseline" approach, both effects appearing for the first time together in the same equations. The more complete formulation given here explicitly describes the relative interplay between wave diffusion and wave dispersion. Various differences found between existing high- and low-order theories were also reconciled within the framework of a broader theory. Some applications were discussed; for example, a previous discussion suggests that Landahl's[8] use of low-order wave models in evaluating the wave focusing and subsequent breakdown of laminar shear flows is valid, even in weakly nonconservative media, provided the waves are almost nondispersive. In Sec. III we extended Whitham's[1] average Lagrangian formalism for fully nonlinear waves in three areas: dissipative effects, modal effects, and high-order effects. The extended formalism allows for frequency and amplitude dispersion and background inhomogeneity as well [Example 3 introduces a nonperiodic mean flow to model spatial nonuniformities, for example, this application not being excluded by the expansion of Eq. (5)]. In several examples, the results obtained were consistent with those obtained by direct multiple-scaling methods; these included local waves, modal waves, and waves on currents (the combined effect of Whitham's "elliptic instability," also known as the "Benjamin-Feir instability," viscous dissipation, and radiation stresses was also cited in one case). These usages were intended to survey the diversity of application.

Some further comment on Eqs. (1) and (2) is appropriate. We have taken the small parameter ϵ in our analysis as being $\epsilon \sim 0(\Omega^i/\Omega^R)$. This usage is similar to that of Stewartson and Stuart[21] where the initial value problem for linearized perturbations is considered for a wave system in plane Poiseuille flow. These authors expand their solution about the critical Reynolds number R_c. Their ϵ, chosen as the linear growth rate, defines new space-time coordinates through the scales ϵt and $\epsilon^{1/2}(x - C^* t)$ where C^* is the linear group velocity. Our Eq. (2) leads to similar results for the wave amplitude equation; the assumption $\epsilon \ll 1$ implicitly assumes some expansion about the neutral curve. Note that Eqs. (1-3) are not centered about K_0, allowing for possible variations in $K(x,t)$. In Ref. 21 and in Example 2, the critical layer initially determines the local dispersion relation and thus fixes the large-scale dynamics; as the wave evolves, the amplitude growth modifies the local modal structure, and this is outside the scope of the present approach.

Finally, observe that kinematic wave theory involves a loss of phase shift information, but this can be partially restored. For example, a general disturbance disperses asymptotically into a train of waves whose properties are known from stationary phase methods; these end conditions can be used as initial conditions for Eqs. (1-3) in a manner similar to "asymptotic matching." The observed "toppling" typical of nonlinear problems cannot be obtained from Sec. II, however: the Stokes correction to frequency, proportional to a^2, does not distinguish between the positive and negative portions of the disturbance wave. In addition, kinematic wave theory lacks "harmonic content"—only one wavenumber can be treated at any one time. Nevertheless, the diversity of application cannot be de-emphasized, this being evidenced by the large number of usages cited.

Application of the present results to "generalized hydraulic jumps" and their stability appears in Ref. 41.

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